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ARCHIMEDEAN L-FACTORS ON $GL(n) \times GL(n)$ AND GENERALIZED BARNES INTEGRALS

BY

ERIC STADE

Department of Mathematics, University of Colorado
Boulder, CO 80309, USA
e-mail: stade@euclid.colorado.edu

ABSTRACT

The Rankin–Selberg method associates, to each local factor $L(s, \pi_\nu \times \pi'_\nu)$ of an automorphic L-function on $GL(n) \times GL(n)$, a certain local integral of Whittaker functions for π_ν and π'_ν. In this paper we show that, if ν is archimedean, and π_ν and π'_ν are spherical principal series representations with trivial central character, then the local L-factor and local integral are, in fact, equal. This result verifies a conjecture of Bump, which predicts that the archimedean situation should, in the present context, parallel the nonarchimedean one.

We also derive, as prerequisite to the above result, some identities for generalized Barnes integrals. In particular, we deduce a new transformation formula for certain single Barnes integrals, and a multiple-integral analog of the classical Barnes' Lemma.

Introduction

Let π and π' be automorphic cuspidal representations of $GL(n, \mathbb{A})$, where $\mathbb{A}$ denotes the adeles over a global field F. One may then define a global L-function $L(s, \pi \times \pi')$, which bears on a number of problems in automorphic forms and representations. For example, suppose π represents a primitive cusp form of weight k and character χ on $\Gamma_0(N)$: Shimura [Shi] has deduced, from the expression of $L(k-1, \pi, \pi')$ (for appropriate π') as a Petersson inner product, the algebraicity of special values of twisted Langlands' L-functions $L(s, \pi, \psi)$. (Here ψ is a primitive Dirichlet character modulo N, and $L(s, \pi, \psi)$ is properly normalized.)

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Another example is the fact — demonstrated by Jacquet [J] in the case $n = 2$, and by Jacquet and Shalika [JS2] in the general case — that the partial L-function $L_S(s, \pi \times \pi')$ (which is closely related to $L(s, \pi \times \pi')$) has a pole at $s = 1$ if and only if π and π' are contragredient representations. This latter result itself has several applications: one such is to growth estimates for Fourier coefficients of Maass forms, cf. the work of Moreno [M] in the case $n = 2$.

The analytic properties of $L(s, \pi \times \pi')$ may be studied by means of the global integral

$$(0.1) \qquad \int_{Z_n(\mathbb{A})GL(n,F)\backslash GL(n,\mathbb{A})} \phi(g)\, \phi'(g)\, E^*(g,s)\, dg;$$

here ϕ and ϕ' are cusp forms in the spaces of π and π' respectively, and $E^*(g, s)$ is an Eisenstein series. ($Z_n(\mathbb{A})$ denotes the center of $GL(n, \mathbb{A})$.) As a function of s, this integral has meromorphic continuation and a functional equation (both inherited from the Eisenstein series). To deduce similar properties of $L(s, \pi \times \pi')$, one writes both this global L-function and the above global integral as products of local factors, and then compares at each place of F the local L-function with the local integral.

More specifically, let us write $\pi = \otimes_\nu \pi_\nu$, with π_ν a representation of $GL(n, F_\nu)$ for each place ν of F (and similarly for π'). Then we have the factorization $L(s, \pi \times \pi') = \prod_\nu L(s, \pi_\nu \times \pi'_\nu)$. Moreover, by the Rankin–Selberg unfolding method, the integral (0.1) may be expressed as $\prod_\nu \Psi(\nu; W, W', f_s)$, where

$$(0.2) \qquad \Psi(\nu; W, W', f_s) = \int_{Z_n(F_\nu)X_n(F_\nu)\backslash GL(n,F_\nu)} W(g)\, W'(g)\, f_s(g)\, dg.$$

Here $X_n(F_\nu) \subset GL(n, F_\nu)$ is the subgroup of upper triangular, unipotent matrices. Also W (resp. W') is a Whittaker function for π_ν (resp. π'_ν), and f_s is in the space of the induced representation

$$\mathrm{Ind}(GL(n, F_\nu), P(n-1, 1, F_\nu), \delta^s),$$

where δ is the modular quasicharacter of the standard parabolic subgroup $P(n-1, 1, F_\nu)$ (with Levi factor $GL(n-1) \times GL(1)$). (We will elaborate on $L(s, \pi_\nu \times \pi'_\nu)$ and $\Psi(\nu; W, W', f_s)$, in the particular case of interest to us, in Section 1 below.) The problem is then to determine the extent to which $\Psi(\nu; W, W', f_s)$ reflects $L(s, \pi_\nu \times \pi'_\nu)$, for each ν.

To this end, Jacquet and Shalika [JS1] show that the factors $L(s, \pi_\nu \times \pi'_\nu)$ and $\Psi(\nu; W, W', f_s)$ are *equal* whenever π_ν and π'_ν are unramified (and non-archimedean), and W and W' are spherical. The more general situation is investigated by Jacquet, Piatetski-Shapiro, and Shalika [JPS] in the case of nonarchimedean ν, and by Jacquet and Shalika [JS3] in the case of archimedean ν. In either case, it is shown that the quotient

$$\Psi(\nu; W, W', f_s)/L(s, \pi_\nu \times \pi'_\nu)$$

has, as a function of s, an analytic continuation, and a functional equation under the simultaneous replacement of s by $1-s$ and of π and π' by their contragredient representations. (In [JPS], [JS1], [JS2], and [JS3], the local integrals in question are of a slightly different form, involving a Schwartz function Φ on F_ν^n. However, the results just stated regarding such integrals apply, by arguments in [GS, pp. 51–56], also to our integrals $\Psi(\nu; W, W', f_s)$.) The investigations of [JPS] and [JS3] actually take place in the broader setting of $GL(n, F_\nu) \times GL(n', F_\nu)$, for n' not necessarily equal to n.

The purpose of the present work is to show that, under certain circumstances, the archimedean places behave like the nonarchimedean ones. Namely, let us return to the case $n' = n$, and assume that π_ν and π'_ν, for ν real or complex, are spherical principal series representations with trivial central characters. (For example, this will be true if π and π' are representations associated with $GL(n, \mathbb{Z})$ Maass forms. See [Bu3] for a discussion.) In particular, π and π' are, by definition (of principal series representation), irreducible. We prove that, under these conditions,

$$\Psi(\nu; W, W', f_s) = L(s, \pi_\nu \times \pi'_\nu), \tag{0.3}$$

for spherical Whittaker functions W and W'. We remark that this result verifies a conjecture of Bump [Bu3], previously known to hold only in the cases $n = 2$ (cf. [J]) and $n = 3$ (cf. [St1]). We also note from [JS3] that, in the situation at hand, the right-hand side of (0.3) is by definition (a factor independent of π_ν and π'_ν, times) a certain product of n^2 gamma functions.

Our result may have applications to generalizations of problems mentioned above. For example, the work of Moreno [M] on Fourier coefficients of $GL(2, \mathbb{Z})$ Maass forms, and that of Shimura [Shi] on special values of $L(s, \pi, \psi)$, both require explicit evaluation of the archimedean integral $\Psi(\nu; W, W', f_s)$ in terms of gamma functions. Higher-rank analogs of these studies would likely require the corresponding formula (0.3).

The starting point for our proof of (0.3) will be the expression of the local integral $\Psi(\nu; W, W', f_s)$ as a (generalized) Barnes integral. By this we mean a (perhaps iterated) complex contour integral of a ratio of products of gamma functions. (See section 2 below for the precise definition.) This expression will allow the application of a new identity — Lemma 2.2, below — for Barnes integrals to the evaluation of $\Psi(\nu; W, W', f_s)$.

Lemma 2.2 is of some independent interest. Indeed this lemma, according to which a certain m-fold Barnes integral reduces to an $(m-1)$-fold integral, may be considered a generalization of "Barnes' (First) Lemma" [Bar1], which is a central result in the theory of Barnes integrals. (We recall Barnes' Lemma in equations (2.3), below.) In addition, our Lemma 2.1 (which is required in the proof of Lemma 2.2) represents a new identity for single Barnes integrals. As such, Lemma 2.1 complements an already extensive list of such identities. For other examples, and for a discussion of Barnes' Lemma and some related formulas, see [Bai].

The connection between Barnes integrals and the archimedean theory of automorphic L-functions was first noticed by Bump, and has also proved fruitful in a context somewhat different from (but parallel to) that of (0.3). Namely, suppose ν is archimedean; let π_ν and π'_ν be spherical principal series representations of $GL(n, F_\nu)$ and $GL(n-1, F_\nu)$ respectively, with trivial central characters. Associated with π_ν and π'_ν is a local L-factor, which is a product of $n(n-1)$ gamma functions, and a local integral analogous to (0.2). (See [JS3] for details.) Bump [Bu2] proved, using "Barnes' Second Lemma" [Bar2] for Barnes integrals, that this local integral and L-factor agree in the case $n = 3$, and conjectured [Bu3] that this should hold for general n. In [St3] we have recently verified this conjecture, also by way of Barnes integrals.

Regarding more general values of (n, n'), it should be noted that the analogous archimedean integral and L-factor will *not* always be equal. This may be seen even in the case $(n, n') = (3, 1)$, cf. [Bu1]. Indeed, heuristic arguments of Bump [Bu3] strongly suggest that such equality will obtain *only* when $|n - n'| \leq 1$.

The present paper will proceed as follows. In Section 1 we state our main theorem (Theorem 1.1), by reformulating (0.3) in language more specific to the situation of interest. Under this reformulation, the integral $\Psi(\nu; W, W', f_s)$ is reinterpreted as a certain integral $\Psi_n(s; a, b)$ of Whittaker functions, where $a, b \in \mathbb{C}^n$ are associated with π_ν and π'_ν respectively. Section 2 contains information concerning Barnes integrals, including the two lemmas mentioned above. In Section 3 we express $\Psi_n(s; a, b)$ as a Barnes integral, by first writing it as a convolution of Mellin transforms of Whittaker functions for $GL(n, F_\nu)$, and then using known formulas for these Mellin transforms as Barnes integrals themselves. We also recall some known cases where such Mellin transforms (or, more exactly, residues thereof) reduce to Mellin transforms of $GL(n-1, F_\nu)$ Whittaker functions. In Section 4 we prove Theorem 1.1, essentially as follows: we apply Lemma 2.2 to our Barnes integral expression for $\Psi_n(s;a,b)$. Examining our integral after this application, we see (by virtue of the residue formulas just mentioned) that we are left with a convolution of Mellin transforms of $GL(n-1,F_\nu)$ Whittaker functions. Our theorem then follows by induction on n (the case $n=2$ being equivalent, as we will see below, to Barnes' Lemma).

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1. Definitions and notation

We will, for the remainder of the paper, restrict our attention to the case $F_\nu = \mathbb{R}$. The case $F_\nu = \mathbb{C}$ is nearly identical, because of the similarity (cf. [St2]) between spherical Whittaker functions on $GL(n,\mathbb{R})$ and those on $GL(n,\mathbb{C})$.

Let $X_n \subset GL(n,\mathbb{R})$ be the group of upper triangular, unipotent matrices, and let $Y_n \subset GL(n,\mathbb{R})$ be the group of diagonal matrices y of the form

$$y = \operatorname{diag}(y_1y_2\cdots y_{n-1}, y_2y_3\cdots y_{n-1}, \ldots, y_{n-1}, 1),$$

with $y_j \in \mathbb{R}^+$ for $1 \le j \le n-1$. Let Z_n denote the center of $GL(n,\mathbb{R})$. Also let $a_1, a_2, \ldots, a_{n-1} \in \mathbb{C}$, and define $a \in \mathbb{C}^n$ by $a = (a_1, a_2, \ldots, a_n)$ where $a_n = -a_1 - a_2 - \cdots - a_{n-1}$. Then the map $\chi_a : X_nY_nZ_n \to \mathbb{C}$ given by

$$\chi_a(xyz) = \prod_{j=1}^{n-1} y_j^{j(n-j)/2} \prod_{k=1}^{j} {y_j}^{a_k}$$

is a quasicharacter of $X_nY_nZ_n$, and may be induced to a representation of $GL(n,\mathbb{R})$. This latter representation, which we denote by $\pi(a)$, will be irreducible for almost all values of a. Assuming, as will will henceforth unless otherwise stated, that a is such a value, $\pi(a)$ is a so-called *spherical principal series representation of* $GL(n,\mathbb{R})$. Note that, by our construction, $\pi(a)$ has trivial central character.

A *spherical Whittaker function* $W_{n,a}(g)$ for $\pi(a)$ is a function on $GL(n,\mathbb{R})$ such that

(a) the restriction, to the space of right translates of $W_{n,a}(g)$, of the regular representation of $GL(n,\mathbb{R})$ is isomorphic to $\pi(a)$;
(b) $W_{n,a}(xg) = e^{2\pi i(x_{1,2}+x_{2,3}+\cdots+x_{n-1,n})}W_{n,a}(g)$ for $x \in X_n$ and $g \in \mathrm{GL}(n,\mathbb{R})$;
(c) $W_{n,a}(gkz) = W_{n,a}(g)$ for $g \in GL(n,\mathbb{R})$, $k \in K_n = O(n,\mathbb{R})$, and $z \in Z_n$;

(d) $W_{n,a}(y)$ is bounded as any $y_j \to \infty$.
Such a function $W_{n,a}(g)$ is known to exist and, by multiplicity-one theorems of Shalika [Sha] and Wallach [Wa], to be unique up to constant multiples.

By the Iwasawa decomposition $GL(n,\mathbb{R}) = X_n Y_n Z_n K_n$, $W_{n,a}(g)$ is determined by its restriction to the subgroup $X_n Y_n$; by property (b) of Whittaker functions, we may write

$$W_{n,a}(xy) = e^{2\pi i(x_{1,2}+x_{2,3}+\cdots+x_{n-1,n})} W_{n,a}(y).$$

The function $W_{n,a}(y)$ is itself often called a "$GL(n,\mathbb{R})$ spherical Whittaker function of type a."

As noted above, $L(2s,\pi(2a)\times\pi(2b))$ is essentially equal to a product of n^2 gamma functions. Moreover, the local integral $\Psi(\mathbb{R}; W_{n,2a}, W_{n,2b}, f_{2s})$ (see (0.2) above), which we will denote more simply by $\Psi_n(s;a,b)$, may be given by
(1.1)

$$\Psi_n(s;a,b) = \Gamma(ns)\int_{(\mathbb{R}^+)^{n-1}} W_{n,2a}(y)\, W_{n,2b}(y) \prod_{j=1}^{n-1} (\pi y_j)^{2js} (2y_j^{-j(n-j)}) \frac{dy_j}{y_j}.$$

(The given normalization of this integral, and of the variables appearing in it, will prove convenient in what follows.) In particular, equation (0.3) amounts, in the context presently under consideration, to the following theorem, which we prove in Section 4 below.

THEOREM 1.1: *Let all notation and assumptions be as above. Then*

$$\Psi_n(s;a,b) = \prod_{j,k=1}^{n} \Gamma(s+a_j+b_k).$$

That is, the local L-factor and its associated local integral are, in this case, the same.

2. Identities for Barnes integrals

We now define, and develop some properties of, Barnes integrals, which will play a significant role in what follows.

A (single) Barnes integral is one of the form

$$\int_z \prod_{j=1}^{M} \Gamma^{\varepsilon_j}(u_j+z) \prod_{k=1}^{N} \Gamma^{\delta_k}(v_k - z)\, dz, \tag{2.1}$$

where M, N are nonnegative integers; $\varepsilon_j, \delta_k = \pm 1$; $u_j, v_k \in \mathbb{C}$. The path of integration is a line parallel to the imaginary axis, indented if necessary to insure that any poles of $\prod_{j=1}^{M} \Gamma^{\varepsilon_j}(u_j + z)$ are to the left of this path, while any poles of $\prod_{k=1}^{N} \Gamma^{\delta_k}(v_k - z)$ are to its right. (Note that poles arise only for those j with $\varepsilon_j = 1$, and those k with $\delta_k = 1$, since the gamma function is never zero. Note also the tacit assumption that, if $\varepsilon_j = \delta_k = 1$, then $u_j + v_k \notin \mathbb{Z}$; otherwise the path of integration could not be chosen as above.) In this paper, unless otherwise specified, *the path of integration of any Barnes integral will always be of this form.*

It may be shown that the integral (2.1) converges absolutely, and uniformly for the real parts of the u_j's and the v_k's in compact subsets of the real line, provided

$$\sum_{j=1}^{M} \varepsilon_j + \sum_{k=1}^{N} \delta_k > 0.$$

Indeed, such convergence follows from "Stirling's formula" (cf. [WW, Section 13.6]):

$$\lim_{|y|\to\infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi} \tag{2.2}$$

(x and y real), uniformly for x in compact subsets. In particular, all (single) Barnes integrals in this paper are, for these reasons, convergent in this way.

Multiple integrals that are of the form (2.1) in each variable of integration are also called Barnes integrals (or, sometimes, *generalized* Barnes integrals). For example, in Section 3 we will express $\Psi_n(s; a, b)$ as such an integral, by writing it as a convolution of the Mellin transforms $\widehat{M_{n,a}}(\cdot)$ and $\widehat{M_{n,b}}(\cdot)$ of $W_{n,2a}(y)$ and $W_{n,2b}(y)$ respectively, and observing that these Mellin transforms are themselves expressible as Barnes integrals.

Stirling's formula (along with, if necessary, induction arguments) may be used to demonstrate the absolute convergence of all multiple Barnes integrals encountered in the present work. Thus our frequent permutations, below, of the orders of integration in such integrals are justified.

Barnes [Bar1] [Bar2] investigated various situations under which integrals like (2.1) reduce to ratios of products gamma functions. One such situation is encapsulated by "Barnes' Lemma" [Bar1], which may be stated as follows:

$$\frac{1}{2\pi i}\int_z \Gamma(\alpha+z)\Gamma(\beta+z)\Gamma(\gamma-z)\Gamma(\delta-z)\,dz = C(\alpha+\gamma, \alpha+\delta, \alpha+\beta+\gamma+\delta) \tag{2.3a}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, where

$$C(u, v, q) = \frac{\Gamma(u)\Gamma(v)\Gamma(q-u)\Gamma(q-v)}{\Gamma(q)}. \tag{2.3b}$$

Our Lemma 2.2 below may be considered as a generalization, to a certain multiple Barnes integral, of Barnes' Lemma.

In order to prove Lemma 2.2, we will first need the following.

LEMMA 2.1: *If $R+r-F-G-H-f-g-h=0$, then*

$$\int_z C(F+z,G+z,R+z)C(f-z,g-z,r-z)\Gamma(H+z)\Gamma(h-z)\,dz$$
$$=\int_z C(F+z,G+z,F+G+h+z)C(f-z,g-z,f+g+H-z)$$
$$\cdot\Gamma(r-f-g+z)\Gamma(R-F-G-z)\,dz.$$

That is, the integral on the left-hand side is invariant under the substitutions

$$(2.5)\quad H\to r-f-g;\quad h\to R-F-G;\quad R\to F+G+h;\quad r\to f+g+H.$$

Proof: Into Barnes' Lemma (2.3a), we put

$$\alpha=t_1;\quad \beta=H;\quad \gamma=h;\quad \delta=t_2.$$

We multiply both sides of the result by

$$\frac{\Gamma(R-F-G+t_1)\Gamma(F-t_1)\Gamma(G-t_1)\Gamma(r-f-g+t_2)\Gamma(f-t_2)\Gamma(g-t_2)}{2\pi i}$$

and integrate in t_1 and t_2, whence

$$(2.4)\qquad \int_z \Gamma(H+z)\Gamma(h-z)$$
$$\cdot\left\{\frac{1}{2\pi i}\int_{t_1}\Gamma(z+t_1)\Gamma(R-F-G+t_1)\Gamma(F-t_1)\Gamma(G-t_1)\,dt_1\right\}$$
$$\cdot\left\{\frac{1}{2\pi i}\int_{t_2}\Gamma(-z+t_2)\Gamma(r-f-g+t_2)\Gamma(f-t_2)\Gamma(g-t_2)\,dt_2\right\}dz$$
$$=\frac{1}{2\pi i}\int_{t_1,t_2}C(t_1+h,t_1+t_2,t_1+H+h+t_2)\Gamma(R-F-G+t_1)$$
$$\cdot\Gamma(F-t_1)\Gamma(G-t_1)\Gamma(r-f-g+t_2)\Gamma(f-t_2)\Gamma(g-t_2)dt_1dt_2$$
$$=\frac{\Gamma(H+h)}{2\pi i}\int_{t_1}\Gamma(R-F-G+t_1)\Gamma(F-t_1)\Gamma(G-t_1)\Gamma(h+t_1)$$
$$\cdot\int_{t_2}\frac{\Gamma(r-f-g+t_2)\Gamma(f-t_2)\Gamma(g-t_2)\Gamma(H+t_2)\Gamma(t_1+t_2)}{\Gamma(H+h+t_1+t_2)}\,dt_2\,dt_1.$$

Here the paths of integration in t_1 and t_2 are as described above for Barnes integrals.

Now note that the left-hand side of (2.4) is, by (2.3a), equal to the integral on the left-hand side of our lemma. So it suffices show that the *right-hand side* of (2.4) is invariant under (2.5). This invariance is clear, since (2.5) takes $H+h$ to $r-f-g+R-F-G=H+h$. So our lemma is proved. ∎

We now prove a "reduction formula" that expresses an m-fold Barnes integral of a certain kind as an $(m-1)$-fold integral. This formula will be central to our proof of Theorem 1.1 (cf. Section 4): in this proof, we will express $\Psi_n(s;a,b)$ in terms of $\Psi_{n-1}(s-(a_1+b_1)/(n-1),a'',b'')$, for appropriate a'',b'', and apply induction on n. We remark that, here and in what follows, a "zero-fold" Barnes integral stands for the integrand itself. (Also, an empty product is understood to equal 1.)

LEMMA 2.2: *Suppose $m\in\mathbb{Z}^+$ and, for $1\le j\le m-1$, $P_j-F_j-G_j=H$ and $p_j-f_j-g_j=h$. Then, for any $L,\ell,z_0\in\mathbb{C}$, we have*

$$\begin{aligned}\int_{z_1,\ldots,z_m}&\left[\prod_{j=1}^{m-1}C(F_j+z_j,G_j+z_j,P_j+z_j+z_{j+1})C(f_j-z_j,g_j-z_j,p_j-z_j-z_{j+1})\right]\\ &\cdot\Gamma(L+z_m)\Gamma(\ell-z_m)\Gamma(H-z_0+z_1)\Gamma(h+z_0-z_1)\,dz_1\cdots dz_m\\ &=2\pi i\int_{z_1,\ldots,z_{m-1}}\left[\prod_{j=1}^{m-1}C(F_j+z_j,G_j+z_j,F_j+G_j+h+z_{j-1}+z_j)\right.\\ &\qquad\left.\cdot C(f_j-z_j,g_j-z_j,f_j+g_j+H-z_{j-1}-z_j)\right]\\ &\qquad\cdot C(h+L+z_{m-1},H+h,H+h+L+\ell)\,dz_1\cdots dz_{m-1}.\end{aligned}$$

Proof: The proof is by induction on m: the case $m=1$ is just Barnes' Lemma (2.3a).

So we assume $m\ge 2$ and note that, by a rearrangement in the order of integration, the quantity on the left-hand side of our lemma equals

(2.6)

$$\begin{aligned}\int_{z_2,\ldots,z_m}&\left[\prod_{j=2}^{m-1}C(F_j+z_j,G_j+z_j,P_j+z_j+z_{j+1})C(f_j-z_j,g_j-z_j,p_j-z_j-z_{j+1})\right]\\ &\cdot\Gamma(L+z_m)\Gamma(\ell-z_m)\left\{\int_{z_1}C(F_1+z_1,G_1+z_1,P_1+z_1+z_2)\right.\\ &\left.\cdot C(f_1-z_1,g_1-z_1,p_1-z_1-z_2)\Gamma(H-z_0+z_1)\Gamma(h+z_0-z_1)\,dz_1\right\}dz_2\cdots dz_m.\end{aligned}$$

After an application of Lemma 2.1, along with the conditions $P_1-F_1-G_1=H$

and $p_1 - f_1 - g_1 = h$, to the quantity in braces, we find that (2.6) equals

(2.7)

$$\int_{z_2,\dots,z_m}\left[\prod_{j=2}^{m-1} C(F_j+z_j, G_j+z_j, P_j+z_j+z_{j+1})C(f_j-z_j, g_j-z_j, p_j-z_j-z_{j+1})\right]$$
$$\cdot\Gamma(L+z_m)\Gamma(\ell-z_m)\left\{\int_{z_1} C(F_1+z_1, G_1+z_1, F_1+G_1+h+z_0+z_1)\right.$$
$$\left.\cdot C(f_1-z_1, g_1-z_1, f_1+g_1+H-z_0-z_1)\Gamma(h+z_1-z_2)\Gamma(H-z_1+z_2)dz_1\right\}$$
$$\cdot dz_2\cdots dz_m$$
$$=\int_{z_1} C(F_1+z_1, G_1+z_1, F_1+G_1+h+z_0+z_1)C(f_1-z_1, g_1-z_1, f_1+g_1+H-z_0-z_1)$$
$$\cdot\left\{\int_{z_2,\dots,z_m}\left[\prod_{j=2}^{m-1} C(F_j+z_j, G_j+z_j, P_j+z_j+z_{j+1})C(f_j-z_j, g_j-z_j, p_j-z_j-z_{j+1})\right]\right.$$
$$\left.\cdot\Gamma(L+z_m)\Gamma(\ell-z_m)\Gamma(H-z_1+z_2)\Gamma(h+z_1-z_2)\,dz_2\cdots dz_m\right\}dz_1$$

(at the last step, we have again permuted the order of integration).

Now note that, by our induction hypothesis, the integral in braces on the right-hand side of (2.7) equals

$$2\pi i\int_{z_2,\dots,z_{m-1}}\left[\prod_{j=2}^{m-1} C(F_j+z_j, G_j+z_j, F_j+G_j+h+z_{j-1}+z_j)\right.$$
$$\left.\cdot C(f_j-z_j, g_j-z_j, f_j+g_j+H-z_{j-1}-z_j)\right]C(h+L+z_{m-1}, H+h, H+h+L+\ell)$$
$$\cdot dz_2\cdots dz_{m-1}.$$

Then the right-hand side of (2.7) itself clearly equals the quantity on the right-hand side of our lemma, and we are done. ∎

3. Mellin transforms of spherical Whittaker functions

We wish to write the integral (1.1) as a convolution of Mellin transforms of $GL(n,\mathbb{R})$ spherical Whittaker functions. To do so, we begin with a definition: if $s \in \mathbb{C}^{n-1}$, then the (normalized) Mellin transform $\widehat{M}_{n,a}(s)$ of $W_{n,2a}(y)$ (cf. Section 1) is given by

$$\text{(3.1a)} \qquad \widehat{M}_{n,a}(s) = \int_{(\mathbb{R}^+)^{n-1}} W_{n,2a}(y)\prod_{j=1}^{n-1}(\pi y_j)^{2s_j}(2y_j{}^{-j(n-j)/2})\frac{dy_j}{y_j}.$$

It is shown in Theorem 2.1 of [St3] that the integral in (3.1a) converges absolutely, and in fact decays exponentially as a function of the $\mathrm{Im}(s_j)$'s, as long as each $\mathrm{Re}(s_j)$ is sufficiently large compared to all the $-\mathrm{Re}(a_k)$'s. (The normalization we have given our Mellin transform is justified by the relative absence of cumbersome powers of 2 or π, in equations (3.2) and (3.3) below.)

The Mellin inversion formula then gives

$$\text{(3.1b)}\qquad W_{n,2a}(y) = \frac{1}{(2\pi i)^{n-1}} \int_{z_1,\cdots,z_{n-1}} \widehat{M}_{n,a}(z) \prod_{j=1}^{n-1} (\pi y_j)^{-2z_j} y_j^{j(n-j)/2}\, dz_j,$$

the path of integration in each z_j being a vertical line in the complex plane, indented if necessary to keep the poles of $\widehat{M}_{n,a}(z)$ on its left.

We now note that $\Psi_n(s;a,b)$ may be expressed as a convolution of $\widehat{M}_{n,a}(\cdot)$ and $\widehat{M}_{n,b}(\cdot)$, as follows. If we define

$$^s z_j = js + z_j \qquad (1 \le j \le n-1),$$

and write

$$z = (z_1, \ldots, z_{n-1}); \quad ^s z = (^s z_1, \ldots, ^s z_{n-1}),$$

then (1.1), (3.1a), and (3.1b) yield

$$\begin{aligned}\text{(3.2)}\qquad &\Psi_n(s;a,b) = \Gamma(ns)\\ &\cdot \int_{(\mathbb{R}^+)^{n-1}} \left\{ \frac{1}{(2\pi i)^{n-1}} \int_{z_1,\cdots,z_{n-1}} \widehat{M}_{n,a}(z) \prod_{j=1}^{n-1} (\pi y_j)^{-2z_j} y_j^{j(n-j)/2}\, dz_j \right\}\\ &\cdot W_{n,2b}(y) \prod_{j=1}^{n-1} (\pi y_j)^{2js} (2 y_j^{-j(n-j)}) \frac{dy_j}{y_j}\\ &= \Gamma(ns) \cdot \frac{1}{(2\pi i)^{n-1}} \int_{z_1,\ldots,z_{n-1}} \widehat{M}_{n,a}(z)\\ &\cdot \left\{ \int_{(\mathbb{R}^+)^{n-1}} W_{n,2b}(y) \prod_{j=1}^{n-1} (\pi y_j)^{2js-2z_j} (2 y_j^{-j(n-j)/2}) \frac{dy_j}{y_j} \right\} dz_1 \cdots dz_{n-1}\\ &= \Gamma(ns) \cdot \frac{1}{(2\pi i)^{n-1}} \int_{z_1,\ldots,z_{n-1}} \widehat{M}_{n,a}(^s z) \widehat{M}_{n,b}(-z)\, dz_1 \cdots dz_{n-1}.\end{aligned}$$

(At the last step, we have substituted $z_j \to {}^s z_j$ for $1 \le j \le n-1$.) The manipulations in (3.2) are formal, but may be justified for $\mathrm{Re}(s)$ sufficienty large (compared to the $-\mathrm{Re}(a_k)$'s and the $-\mathrm{Re}(b_k)$'s) and for appropriate contours of integration in the z_j's. The end result of these manipulations holds, by analytic continuation arguments, as long as these contours separate the poles of $\widehat{M}_{n,a}({}^s z)$ from those of $\widehat{M}_{n,b}(-z)$.

Equation (3.2) will now allow us to apply known facts concerning $\widehat{M}_{n,a}(s)$ to the evaluation of $\Psi_n(s;a,b)$. We summarize the facts we will need: all of these come from [St3].

We begin by writing $\widehat{M}_{n,a}(s)$ as a contour integral involving a lower-rank Mellin transform. Namely, Theorem 2.1 in [St3] gives (if $C(u,v,q)$ is as in (2.3b)):

(3.3a)
$$\widehat{M}_{n,a}(s) = \Gamma(s_1+a_1)\Gamma(s_{n-1}-a_1)\cdot\frac{1}{(2\pi i)^{n-3}}\int_{t_1,\ldots,t_{n-3}}\widehat{M}_{n-2,\widehat{a}}(\widehat{t}\,)$$
$$\cdot\left[\prod_{j=1}^{n-2}C(t_j+s_j,t_{j-1}+a_2+s_j,t_{j-1}+t_j+a_1+a_2+s_j+s_{j+1})\right]dt_1\cdots dt_{n-3},$$

where

(3.3b)
$$\widehat{t} = \left(-t_j - \frac{j(a_1+a_2)}{n-2}\right)_{1\le j\le n-3};$$

(3.3c)
$$\widehat{a} = (\widehat{a}_j)_{1\le j\le n-2} = \left(a_{j+2} + \frac{a_1+a_2}{n-2}\right)_{1\le j\le n-2};$$

(3.3d)
$$t_0 = 0; \quad t_{n-2} = -a_1 - a_2.$$

(Our notation here is somewhat different than that of [St3]: what we called c there is denoted by $\widehat{a}$ here. Also, the product on j in the integrand here is a re-ordering of the one appearing there.) The paths of integration are vertical lines, indented if necessary so that all poles of $\widehat{M}_{n-2,\widehat{a}}(\widehat{t})$ (considered as a function of the t_j's) are to the right of these paths, while all poles of the product on j (also as a function of the t_j's) are to their left. (In [St3], we consider only paths that are vertical lines *per se*; for this to be possible the s_j's must have real parts sufficiently large compared to those of the $-a_k$'s.)

In order that the right-hand side of (3.3a) make sense in the cases $n=2$ and $n=3$, we define
$$\widehat{M}_{0,a}(s) = \widehat{M}_{1,a}(s) = 1.$$

We also understand the factor $(2\pi i)^{3-n}$, and the contour of integration, to appear only if $n-3>0$. In particular, (3.3a) then yields

(3.4)
$$\widehat{M}_{2,a}(s) = \Gamma(s+a)\Gamma(s-a)$$

where, by an abuse of notation, $s = s_1$ and $a = (a_1, a_2) = (a, -a)$. This formula is well-known, cf. [Bu1].

We may now write $\Psi_n(s; a, b)$ as a multiple Barnes integral. Namely, by combining (3.2) with (3.3a), we get

$$\tag{3.5} \begin{aligned} \Psi_n(s;a,b) &= \Gamma(ns) \\ &\cdot \frac{1}{(2\pi i)^{n-1}} \int_{z_1,\ldots,z_{n-1}} \Gamma({}^s z_1 + a_1)\Gamma({}^s z_{n-1} - a_1)\Gamma(-z_1 + b_1)\Gamma(-z_{n-1} - b_1) \\ &\cdot \left\{ \frac{1}{(2\pi i)^{n-3}} \int_{t_1,\ldots,t_{n-3}} \widehat{M}_{n-2,\widehat{a}}(\widehat{t}) \right. \\ &\cdot \left. \left[\prod_{j=1}^{n-2} C(t_j + {}^s z_j, t_{j-1} + a_2 + {}^s z_j, t_{j-1} + t_j + a_1 + a_2 + {}^s z_j + {}^s z_{j+1}) \right] dt_1 \cdots dt_{n-3} \right\} \\ &\cdot \left\{ \frac{1}{(2\pi i)^{n-3}} \int_{x_1,\ldots,x_{n-3}} \widehat{M}_{n-2,\widehat{b}}(\widehat{x}) \right. \\ &\cdot \left. \left[\prod_{j=1}^{n-2} C(x_j - z_j, x_{j-1} + b_2 - z_j, x_{j-1} + x_j + b_1 + b_2 - z_j - z_{j+1}) \right] dx_1 \cdots dx_{n-3} \right\} \\ &\cdot dz_1 \cdots dz_{n-1}, \end{aligned}$$

where $\widehat{x}$ bears the same relation to the x_j's and to b as $\widehat{t}$ does to the t_j's and to a (cf. (3.3b)); $\widehat{b}$ bears the same relation to b as $\widehat{a}$ does to a (cf. (3.3c)); $x_0 = 0$; and $x_{n-2} = -b_1 - b_2$.

Our proof below of Theorem 1.1 will also require some information, which we now recall, regarding residues of $\widehat{M}_{n,a}(s)$. Specifically, we have

$$\tag{3.6a} \lim_{s_1 \to -a_1} (s_1 + a_1)\widehat{M}_{n,a}(s) = \left[\prod_{j=2}^{n} \Gamma(a_j - a_1) \right] \widehat{M}_{n-1,a''}(s''),$$

where

$$\tag{3.6b} a'' = \left(a_{j+1} + \frac{a_1}{n-1} \right)_{1 \le j \le n-1}; \quad s'' = \left(s_{j+1} + \frac{n-1-j}{n-1} a_1 \right)_{1 \le j \le n-2}.$$

Indeed, equations (3.6) are just the case $k = 1$ of Theorem 3.2 in [St3].

For our computations below, it will in fact be useful to rewrite (3.6a), using equations (3.3). To do so we note that, on the right-hand side of (3.3a), the variable s_1 appears only in $\Gamma(s_1 + a_1)$ and in the $j = 1$ factor in the product. Since the gamma function has a simple pole, of residue one, at zero (and since, again, $t_0 = 0$), we compute from (2.3b) that

$$\tag{3.7} \lim_{s_1 \to -a_1} (s_1 + a_1)\Gamma(s_1 + a_1)C(t_1 + s_1, t_0 + a_2 + s_1, t_0 + t_1 + a_1 + a_2 + s_1 + s_2)$$

$$= \frac{\Gamma(t_1 - a_1)\Gamma(a_2 - a_1)\Gamma(s_2 + a_1 + a_2)\Gamma(t_1 + s_2 + a_1)}{\Gamma(t_1 + s_2 + a_2)}.$$

Combining (3.3a), (3.6a), and (3.7), we get

$$\begin{aligned}(3.8)\qquad & \Gamma(s_{n-1} - a_1)\Gamma(s_2 + a_1 + a_2) \\ & \cdot \frac{1}{(2\pi i)^{n-3}} \int_{t_1,\ldots,t_{n-3}} \widehat{M}_{n-2,\widehat{a}}(\widehat{t}) \frac{\Gamma(t_1 - a_1)\Gamma(t_1 + s_2 + a_1)}{\Gamma(t_1 + s_2 + a_2)} \\ & \cdot \left[\prod_{j=2}^{n-2} C(t_j + s_j, t_{j-1} + a_2 + s_j, t_{j-1} + t_j + a_1 + a_2 + s_j + s_{j+1})\right] dt_1 \cdots dt_{n-3} \\ & = \left[\prod_{j=3}^{n} \Gamma(a_j - a_1)\right] \widehat{M}_{n-1,a''}(s''),\end{aligned}$$

with a'' and s'' as in (3.6b). We remark that the interchange of the limit in (3.6a) with the integral in (3.3a) is valid, for appropriate choice of the contours of integration, *provided* the poles in t_1 of $\Gamma(t_1 - a_1)$ are disjoint from those of $\widehat{M}_{n-2,\widehat{a}}(\widehat{t})$. But, as is shown in [St3], the latter has its leftmost pole at $t_1 = \min\{a_3, a_4, \cdots a_n\}$. So we need only assume that $a_1 < \min\{a_3, a_4, \ldots, a_n\}$. Analytic continuation arguments insure that we do not, in our application below of (3.8), suffer any loss of generality by making this assumption. (Strictly speaking, we are assuming that a is such that $\pi(a)$ is irreducible. However, the Whittaker function $W_{n,2a}$ may in fact be extended to a holomorphic function of a. So, in applying the analytic continuation arguments just mentioned, we needn't worry about the measure-zero set of values of a where $\pi(a)$ is reducible.)

4. Proof of Theorem 1.1

We proceed by induction on n: the case $n = 2$ follows directly from formula (3.2) for $\Psi_n(s; a, b)$, formula (3.4) for $\widehat{M}_{2,a}(s)$, and Barnes' Lemma (2.3a).

So we assume $n > 2$, and consider (3.5). We change the order of integration, and recall that ${}^s z_j = z_j + js$, to get

$$\begin{aligned}(4.1)\qquad & \Psi_n(s; a, b) = \Gamma(ns) \cdot \frac{1}{(2\pi i)^{n-3}} \int_{t_1,\ldots,t_{n-3}} \widehat{M}_{n-2,\widehat{a}}(\widehat{t}\,) \\ & \cdot \frac{1}{(2\pi i)^{n-3}} \int_{x_1,\ldots,x_{n-3}} \widehat{M}_{n-2,\widehat{b}}(\widehat{x}\,) \cdot \left\{ \frac{1}{(2\pi i)^{n-1}} \int_{z_1,\ldots,z_{n-1}} \right. \\ & \cdot \left[\prod_{j=1}^{n-2} C(t_j + js + z_j, t_{j-1} + a_2 + js + z_j, t_{j-1} + t_j + a_1 + a_2 + (2j+1)s + z_j + z_{j+1})\right.\end{aligned}$$

$$\left.\cdot C(x_j - z_j, x_{j-1} + b_2 - z_j, x_{j-1} + x_j + b_1 + b_2 - z_j - z_{j+1})\right]$$

$$\left.\cdot\Gamma(z_{n-1} + (n-1)s - a_1)\Gamma(-z_{n-1} - b_1)\Gamma(z_1 + s + a_1)\Gamma(-z_1 + b_1)\, dz_1 \cdots dz_{n-1}\right\}$$

$$\cdot dt_1 \cdots dt_{n-3}\, dx_1 \cdots dx_{n-3}.$$

If we define $z_0 = 0$, then we may apply Lemma 2.2, with $m = n-1$, to the integral in braces; (4.1) then reads

$$\Psi_n(s; a, b) = \Gamma(ns) \cdot \frac{1}{(2\pi i)^{n-3}} \int_{t_1,\ldots,t_{n-3}} \widehat{M}_{n-2,\widehat{a}}(\widehat{t}\,) \tag{4.2}$$

$$\cdot \frac{1}{(2\pi i)^{n-3}} \int_{x_1,\ldots,x_{n-3}} \widehat{M}_{n-2,\widehat{b}}(\widehat{x}\,)\left\{\frac{1}{(2\pi i)^{n-2}} \int_{z_1,\ldots,z_{n-2}}\right.$$

$$\cdot\left[\prod_{j=1}^{n-2} C(t_j + js + z_j, t_{j-1} + a_2 + js + z_j, t_{j-1} + t_j + a_2 + b_1 + 2js + z_{j-1} + z_j)\right.$$

$$\left.\cdot C(x_j - z_j, x_{j-1} + b_2 - z_j, x_{j-1} + x_j + a_1 + b_2 + s - z_{j-1} - z_j)\right]$$

$$\left.\cdot C((n-1)s - a_1 + b_1 + z_{n-2}, s + a_1 + b_1, ns)\, dz_1 \cdots dz_{n-2}\right\} dt_1 \cdots dt_{n-3}\, dx_1 \cdots dx_{n-3}.$$

Changing once more the order of integration, and again using the fact that ${}^s z_j = js + z_j$, we find from (4.2) that

$$\Psi_n(s; a, b) = \Gamma(s + a_1 + b_1)\Gamma((n-1)s - a_1 - b_1) \tag{4.3}$$

$$\cdot \frac{1}{(2\pi i)^{n-2}} \int_{z_1,\ldots,z_{n-2}} \left\{ \frac{1}{(2\pi i)^{n-3}} \int_{t_1,\ldots,t_{n-3}} \widehat{M}_{n-2,\widehat{a}}(\widehat{t}\,)\Gamma(-a_1 + b_1 + s + {}^s z_{n-2})\right.$$

$$\left.\cdot\left[\prod_{j=1}^{n-2} C(t_j + {}^s z_j, t_{j-1} + a_2 + {}^s z_j, t_{j-1} + t_j + a_2 + b_1 + s + {}^s z_{j-1} + {}^s z_j)\right] dt_1 \cdots dt_{n-3}\right\}$$

$$\cdot\left\{ \frac{1}{(2\pi i)^{n-3}} \int_{x_1,\ldots,x_{n-3}} \widehat{M}_{n-2,\widehat{b}}(\widehat{x}\,)\Gamma(a_1 - b_1 + s - z_{n-2})\right.$$

$$\left.\cdot\left[\prod_{j=1}^{n-2} C(x_j - z_j, x_{j-1} + b_2 - z_j, x_{j-1} + x_j + a_1 + b_2 + s - z_{j-1} - z_j)\right] dx_1 \cdots dx_{n-3}\right\}$$

$$\cdot dz_1 \cdots dz_{n-2}.$$

Let us examine carefully the integral in the t_j's, in (4.3). If we define

$$a_1^* = -b_1 - s + \frac{a_1 + b_1 + s}{n}; \quad a_j^* = a_j + \frac{a_1 + b_1 + s}{n} \quad (2 \le j \le n);$$

$$s_j^* = b_1 + s + {}^s z_{j-1} - j\left(\frac{a_1 + b_1 + s}{n}\right) \quad (2 \le j \le n-1),$$

and make the change of variable

$$t_j^* = t_j + j\left(\frac{a_1+b_1+s}{n}\right)$$

for $1 \le j \le n-3$, then (since $(a_1+b_1+s)/n = (a_1+a_2-a_1^*-a_2^*)/(n-2)$) we find that

$$(4.4)\qquad \frac{1}{(2\pi i)^{n-3}}\int_{t_1,\ldots,t_{n-3}} \widehat{M}_{n-2,\widehat{a}}(\widehat{t}\,)\Gamma(-a_1+b_1+s+{}^sz_{n-2})$$
$$\cdot\left[\prod_{j=1}^{n-2} C(t_j+{}^sz_j, t_{j-1}+a_2+{}^sz_j, t_{j-1}+t_j+a_2+b_1+s+{}^sz_{j-1}+{}^sz_j)\right]dt_1\cdots dt_{n-3}$$
$$= \Gamma(s+a_2+b_1)\Gamma(s_{n-1}^*-a_1^*)\Gamma(s_2^*+a_1^*+a_2^*)$$
$$\cdot\frac{1}{(2\pi i)^{n-3}}\int_{t_1^*,\ldots,t_{n-3}^*} \widehat{M}_{n-2,\widehat{a^*}}(\widehat{t^*})\frac{\Gamma(t_1^*-a_1^*)\Gamma(t_1^*+s_2^*+a_1^*)}{\Gamma(t_1^*+s_2^*+a_2^*)}$$
$$\cdot\left[\prod_{j=2}^{n-2} C(t_j^*+s_j^*, t_{j-1}^*+a_2^*+s_j^*, t_{j-1}^*+t_j^*+a_1^*+a_2^*+s_j^*+s_{j+1}^*)\right]dt_1^*\cdots dt_{n-3}^*$$

where $\widehat{t^*}$ bears the same relation to the t_j^*'s and to a^* as $\widehat{t}$ does to the t_j's and to a (cf. (3.3b)); $\widehat{a^*}$ bears the same relation to a^* as $\widehat{a}$ does to a (cf. (3.3c)); $t_0^*=0$; and $t_{n-2}^* = -a_1^*-a_2^*$. (We have also used the fact that $C(u,v,q)=C(q-u,q-v,q)$, cf. the definition (2.3b).) Comparing (4.4) and (3.8), we conclude that

$$(4.5)\qquad \frac{1}{(2\pi i)^{n-3}}\int_{t_1,\ldots,t_{n-3}} \widehat{M}_{n-2,\widehat{a}}(\widehat{t})\Gamma(-a_1+b_1+s+{}^sz_{n-2})$$
$$\cdot\left[\prod_{j=1}^{n-2} C(t_j+{}^sz_j, t_{j-1}+a_2+{}^sz_j, t_{j-1}+t_j+a_2+b_1+s+{}^sz_{j-1}+{}^sz_j)\right]dt_1\cdots dt_{n-3}$$
$$= \Gamma(s+a_2+b_1)\left[\prod_{j=3}^{n}\Gamma(a_j^*-a_1^*)\right]\widehat{M}_{n-1,(a^*)''}((s^*)'')$$
$$= \left[\prod_{j=2}^{n}\Gamma(s+a_j+b_1)\right]\widehat{M}_{n-1,a''}((z_j+j(s-a_1/(n-1)))_{1\le j\le n-2}),$$

since a bit of algebra shows that

$$a_j^*-a_1^* = s+a_j+b_1 \quad (3\le j\le n);$$
$$(a^*)'' = \left(a_{j+1}^*+\frac{a_1^*}{n-1}\right)_{1\le j\le n-1} = \left(a_{j+1}+\frac{a_1}{n-1}\right)_{1\le j\le n-1} = a'';$$
$$(s^*)'' = \left(s_{j+1}^*+\frac{n-1-j}{n-1}a_1^*\right)_{1\le j\le n-2} = \left(z_j+js-\frac{j}{n-1}a_1\right)_{1\le j\le n-2}.$$

Now to evaluate the integral in the x_j's, in (4.3), we may simply substitute

$$t_j \to x_j; \quad b_j \leftrightarrow a_j; \quad z_j \leftrightarrow -^s z_j$$

(for all relevant j) into (4.5). The result is

$$(4.6) \qquad \frac{1}{(2\pi i)^{n-3}} \int_{x_1,\ldots,x_{n-3}} \widehat{M}_{n-2,\widehat{b}}(\widehat{x})\Gamma(a_1 - b_1 + s - z_{n-2})$$
$$\cdot\left[\prod_{j=1}^{n-2} C(x_j - z_j, x_{j-1} + b_2 - z_j, x_{j-1} + x_j + a_1 + b_2 + s - z_{j-1} - z_j)\right] dx_1 \cdots dx_{n-3}$$
$$= \left[\prod_{j=2}^{n} \Gamma(s + a_1 + b_j)\right] \widehat{M}_{n-1,b''}((-z_j - jb_1/(n-1)))_{1\le j\le n-2}).$$

Combining (4.3), (4.5), and (4.6) then yields

$$(4.7) \qquad \Psi_n(s; a, b)$$
$$= \Gamma(s + a_1 + b_1)\Gamma((n-1)s - a_1 - b_1)\left[\prod_{j=2}^{n} \Gamma(s + a_j + b_1)\Gamma(s + a_1 + b_j)\right]$$
$$\cdot \frac{1}{(2\pi i)^{n-2}} \int_{z_1,\ldots,z_{n-2}} \widehat{M}_{n-1,a''}((z_j + j(s - a_1/(n-1))))_{1\le j\le n-2})$$
$$\cdot \widehat{M}_{n-1,b''}((-z_j - jb_1/(n-1)))_{1\le j\le n-2})\, dz_1 \cdots dz_{n-2}$$
$$= \Gamma(s + a_1 + b_1)\left[\prod_{j=2}^{n} \Gamma(s + a_j + b_1)\Gamma(s + a_1 + b_j)\right] \Psi_{n-1}\left(s - \frac{a_1 + b_1}{n-1}, a'', b''\right);$$

at the last step, we have substituted $z_j \to z_j - jb_1/(n-1)$ for $1 \le j \le n-2$, and applied (3.2). By our induction hypothesis, (4.7) reads

$$\Psi_n(s; a, b) = \Gamma(s + a_1 + b_1)$$
$$\cdot\left[\prod_{j=2}^{n} \Gamma(s + a_j + b_1)\Gamma(s + a_1 + b_j)\right] \prod_{j,k=1}^{n-1} \Gamma\left(s - \frac{a_1 + b_1}{n-1} + a_j'' + b_k''\right)$$
$$= \Gamma(s + a_1 + b_1)\left[\prod_{j=2}^{n} \Gamma(s + a_j + b_1)\Gamma(s + a_1 + b_j)\right] \prod_{j,k=2}^{n} \Gamma(s + a_j + b_k).$$

But this is precisely the statement of Theorem 1.1, and we are done. ∎

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